Georgia Assessments for the Certification of Educators®

GACE® Study Companion

Mathematics Assessment

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About the Assessment

Assessment Name	Mathematics
Grade Level	6–12
Test Code	Test I: 022 Test II: 023 Combined Test I and Test II: 522
Testing Time	Test I: 2 hours 15 minutes Test II: 2 hours 15 minutes Combined Test I and Test II: 4.5 hours
Test Duration	Test I: 2 hours 30 minutes Test II: 2 hours 30 minutes Combined Test I and Test II: 5 hours
Test Format	Computer delivered
Number of Selected-response Questions	Test I: 50 Test II: 50 Combined Test I and Test II: 100
Question Format	The test consists of a variety of short-answer questions such as selected-response questions, where you select one answer choice or multiple answer choices (depending on what the question asks for), questions where you enter your answer in a text box, and other types of questions. You can review the possible question types in the ***Guide to Taking a GACE Computer-delivered Test***.
Number of Constructed-response Questions	Test I: 0 Test II: 0 Combined Test I and Test II: 0

The GACE Mathematics assessment is designed to measure the professional knowledge of prospective teachers of 6–12 mathematics in the state of Georgia.

This assessment includes two tests. You may take either test individually or the full assessment in a single session. The testing time is the amount of time you will have to answer the questions on the test. Test duration includes time for tutorials and directional screens that may be included in the test.

The questions in this assessment assess both basic knowledge across content areas and the ability to apply principles.

The total number of questions that are scored is typically smaller than the total number of questions on the test. Most tests that contain selected-response questions also include embedded pretest questions, which are not used in calculating your score. By including pretest questions in the assessment, ETS is able to analyze actual test-taker performance on proposed new questions and determine whether they should be included in future versions of the test.

Content Specifications

Each test in this assessment is organized into content **subareas**. Each subarea is further defined by a set of **objectives** and their **knowledge statements**.

- The objectives broadly define what an entry-level educator in this field in Georgia public schools should know and be able to do.
- The knowledge statements describe in greater detail the knowledge and skills eligible for testing.
- Some tests also include content material at the evidence level. This content serves as descriptors of what each knowledge statement encompasses.

See a breakdown of the subareas and objectives for the tests in this assessment on the following pages.

Test I Subareas

Subarea	Approx. Percentage of Test
I. Number and Quantity	30%
II. Algebra	40%
III. Discrete Mathematics and Calculus	30%

Test I Objectives

Subarea I: Number and Quantity

Objective 1: Understands and applies knowledge of the real number system and vector and matrix quantities

The beginning Mathematics teacher:

A. Understands the properties of exponents
  - Performs operations involving exponents, including negative and rational exponents
  - Demonstrates an understanding of the properties of exponential expressions
  - Uses the properties of exponents to rewrite expressions that have radicals or rational exponents

B. Understands the properties of rational and irrational numbers and the interactions between those sets of numbers
  - Recognizes that the sum or product of two rational numbers is rational
  - Recognizes that the sum of a rational number and an irrational number is irrational
  - Recognizes that the product of a nonzero rational number and an irrational number is irrational
  - Recognizes that the sum or product of two irrational numbers can be rational or irrational

C. Is familiar with the representation and modeling of vector quantities and how operations on vectors are performed
  - Represents vector quantities by directed line segments and uses appropriate symbols for vectors and their magnitudes
  - Finds the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point
  - Solves problems involving velocity and other quantities that can be represented by vectors

- Adds vectors end-to-end, component-wise, and by the parallelogram rule
- Given two vectors in magnitude and direction form, determines the magnitude and direction of their sum

D. Understands how to perform operations on matrices and how to use matrices in applications

- Uses matrices to represent and manipulate data
- Multiplies matrices by scalars to produce new matrices
- Adds, subtracts, and multiplies matrices of appropriate dimensions
- Understands that matrix multiplication for square matrices is not a commutative operation but still satisfies the associative and distributive properties
- Understands the role played by zero and identity matrices in matrix addition and multiplication
- Understands that the determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse

E. Understands how to solve problems involving ratios, proportions, averages, percents, and metric and traditional unit conversions

- Applies the concept of a ratio and uses ratio language and notation to describe a relationship between two quantities and solve problems
- Uses ratio reasoning to convert rates
- Solves problems involving scale factors
- Recognizes and represents proportional and inversely proportional relationships between two quantities
- Uses proportional relationships to solve multistep ratio, average, and percent problems
- Solves measurement and estimation problems involving time, length, temperature, volume, and mass in both the U.S. customary system and the metric system, where appropriate
- Converts units within the metric and customary systems

F. Understands various ways to represent, compare, estimate, and perform calculations on very large and very small numbers; e.g., scientific notation, orders of magnitude

- Represents and compares very large and very small numbers
- Uses orders of magnitude to estimate very large and very small numbers
- Performs calculations on numbers in scientific notation

Objective 2: Understands and applies knowledge of quantities and the complex number system

The beginning Mathematics teacher:

A. Understands how to solve problems by reasoning quantitatively; e.g., dimensional analysis, reasonableness of solutions

    - Uses units as a way to understand problems and to guide the solution of multistep problems
    - Chooses and interprets units consistently in formulas
    - Chooses and interprets the scale and the origin in graphs and data displays
    - Recognizes the reasonableness of results within the context of a given problem

B. Understands the structure of the natural, integer, rational, real, and complex number systems and how the basic operations (+, –, ×, and ÷) on numbers in these systems are performed

    - Solves problems using addition, subtraction, multiplication, and division of rational, irrational, and complex numbers
    - Applies the order of operations
    - Given operations on a number system, determines whether the properties (e.g., commutative, associative, distributive) hold
    - Compares, classifies, and orders real numbers
    - Demonstrates an understanding of the properties of counting numbers; e.g., prime, composite, prime factorization, even, odd, factors, multiples

C. Knows how complex numbers and operations on them are represented in the complex plane

    - Represents complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers)
    - Explains why the rectangular and polar forms of a given complex number represent the same number
    - Represents addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane, and uses properties of the representation for computation
    - Calculates the distance between numbers in the complex plane as the modulus of the difference, and the midpoint of a segment as the average of the numbers at its endpoints

D. Understands how to work with complex numbers when solving polynomial equations and rewriting polynomial expressions

    - Solves quadratic equations with real coefficients that have complex solutions
    - Extends polynomial identities to the complex numbers; e.g., $x^2 + y^2 = (x + yi)(x - yi)$

E. Knows how to analyze both precision and accuracy in measurement situations

- Chooses a level of accuracy appropriate to limitations on measurement when reporting quantities
- Calculates or estimates absolute and relative error in the numerical answer to a problem

Subarea II: Algebra

Objective 1: Sees structure in expressions and understands arithmetic with polynomials and rational expressions

The beginning Mathematics teacher:

A. Understands how to write algebraic expressions in equivalent forms

- Uses the structure of an expression to identify ways to rewrite it
- Understands how to rewrite quadratic expressions for specific purposes; e.g., factoring/finding zeros, completing the square/finding maxima or minima
- Uses the properties of exponents to rewrite expressions for exponential functions

B. Understands how to perform arithmetic operations on polynomials

- Adds, subtracts, multiplies, and divides polynomials

C. Understands the relationship between zeros of polynomial functions (including their graphical representation) and factors of the related polynomial expressions

- Knows and applies the remainder theorem: for a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$, so $p(a) = 0$ if and only if $x - a$ is a factor of $p(x)$
- Uses factorization to identify zeros of polynomials
- Uses zeros of a polynomial to construct a rough graph of the function defined by the polynomial

D. Understands how to use the binomial theorem to solve problems

- Applies the binomial theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n

E. Understands how to rewrite rational expressions and perform arithmetic operations on rational expressions

- Rewrites simple rational expressions in different forms
- Understands that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression
- Adds, subtracts, multiplies, and divides rational expressions

F. Understands the properties of number systems under various operations

- Given operations on algebraic expressions, determines whether properties (e.g., commutative, associative, distributive) hold
- Performs calculations using newly defined functions

Objective 2: Understands how to create equations and how to reason with equations and inequalities

The beginning Mathematics teacher:

A. Understands how to create equations and inequalities that describe relationships

- Creates equations and inequalities in one variable and uses them to solve problems and graph solutions on the number line
- Creates equations and inequalities to represent relationships between quantities, solves problems, and graphs them on the coordinate plane with labels and scales
- Represents constraints by equations, inequalities, or systems of equations and/or inequalities, and interprets solutions as viable or nonviable options in a modeling context
- Rearranges formulas to highlight a quantity of interest; e.g., solve $d = rt$ for t

B. Understands how to justify the reasoning process used to solve equations, including analysis of potential extraneous solutions

- States each step in solving a simple equation
- Solves simple rational and radical equations in one variable, incorporating analysis of possible extraneous solutions

C. Understands how varied techniques (e.g., graphical, algebraic) are used to solve equations and inequalities

- Solves linear equations and inequalities, including equations with coefficients represented by letters
- Uses the method of completing the square to transform any quadratic equation in x into the equivalent form $(x - p)^2 = q$
- Solves equations using a variety of methods (e.g., using graphs, using the quadratic formula, factoring)
- Uses different methods (e.g., discriminant analysis, graphical analysis) to determine the nature of the solutions of a quadratic equation

D. Understands how varied techniques (e.g., graphical, algebraic, matrix) are used to solve systems of equations and inequalities

- Explains why, when solving a system of two equations using the elimination method, replacing one or both equations with a scalar multiple produces a system with the same solutions as the solutions of the original system

- Solves a system consisting of two linear equations in two variables algebraically and graphically
- Solves a system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically
- Represents a system of linear equations as a single matrix equation
- Finds the inverse of a matrix if it exists and uses it to solve systems of linear equations
- Explains why the x-coordinates of the intersection points of the graphs of $y = f(x)$ and $y = g(x)$ are the solutions of $f(x) = g(x)$
- Finds the solutions of $f(x) = g(x)$ approximately (e.g., uses technology to graph the functions, makes tables of values, finds successive approximations); includes cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, or logarithmic functions
- Graphs the solutions to a linear inequality in two variables as a half-plane (excluding the boundary in the case of a strict inequality), and graphs the solution set to a system of linear inequalities in two variables as the intersection of the corresponding half-planes

E. Understands the concept of rate of change of nonlinear functions

- Calculates and interprets the average rate of change of a function presented symbolically, numerically, or graphically over a specified interval

F. Understands the concepts of intercept(s) of a line and slope as a rate of change

- Calculates and interprets the intercepts of a line
- Calculates and interprets the slope of a line presented symbolically, numerically, or graphically
- Estimates the rate of change of a linear function from a graph

G. Understands how to find the zero(s) of functions

- Uses a variety of techniques to find and analyze the zero(s) (real and complex) of functions

Subarea III: Discrete Mathematics and Calculus

Objective 1: Understands and applies knowledge of discrete mathematics

The beginning Mathematics teacher:

A. Understands sequences; e.g., arithmetic, recursively defined, geometric

- Writes arithmetic and geometric sequences both recursively and with an explicit formula, uses them to model situations, and translates between the two forms
- Evaluates, extends, or algebraically represents rules that involve number patterns
- Explores patterns in order to make conjectures, predictions, or generalizations

B. Understands the differences between discrete and continuous representations (e.g., data, functions) and how each can be used to model various phenomena

- Understands the differences between discrete and continuous representations; e.g., data, functions
- Understands how discrete and continuous representations can be used to model various phenomena

C. Knows how to model and solve problems using vertex-edge graphs, trees, and networks

- Constructs, uses, and interprets simple diagrams to solve problems
- Solves linear programming problems

D. Understands basic terminology and symbols of logic

- Understands the basic terminology of logic
- Uses logic to evaluate the truth of statements
- Uses logic to evaluate the equivalence of statements; e.g., statement and contrapositive
- Identifies basic properties of quantifiers; e.g., for all, there exists
- Negates statements involving quantifiers; e.g., for all, there exists

E. Understands how to use counting techniques such as the multiplication principle, permutations, and combinations

- Uses counting techniques to solve problems

F. Understands basic set theory; e.g., unions, differences, and Venn diagrams

- Solves problems using basic set theory; i.e., union, intersection, complement, difference
- Uses Venn diagrams to answer questions about sets

Objective 2: Understands calculus concepts and applies knowledge to solve calculus problems

The beginning Mathematics teacher:

A. Understands the meaning of a limit of a function and how to calculate limits of functions, how to determine when the limit does not exist, and how to solve problems using the properties of limits

- Graphically analyzes the limit of $f(x)$ as x approaches a fixed value from both left and right
- Solves limit problems (e.g., a constant times a function, the sum of two functions, the product and quotient of two functions) using properties of limits, where all limits of the individual functions exist at the value that x is approaching
- Analyzes one-sided limits for various functions to see whether or not the limit exists
- Recognizes limits that do not exist, such as $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ and $\lim_{x\to 0} \frac{1}{\sqrt[3]{x^2}}$

B. Understands the derivative of a function as a limit, as the slope of a line tangent to a curve, and as a rate of change

- Constructs a function graph for a given function and a given point (a, $f(a)$), and explains what happens to the succession of slopes of secant lines connecting (a, $f(a)$) to (x, $f(x)$) as x approaches a, from both the right side and the left side
- Uses the limit definition of the derivative to find the derivative of a given function at a given value of x and to find the derivative function

C. Understands how to show that a particular function is continuous

- Applies the three steps (i.e, $f(a)$ exists, $\lim_{x \to a} f(x)$ exists, and $f(a) = \lim_{x \to a} f(x)$) that are part of the definition of what it means for a function to be continuous at $x = a$ to verify whether a given function is continuous at a given point

D. Knows the relationship between continuity and differentiability

- Gives examples of functions that are continuous at $x = a$ but not differentiable at $x = a$, and explains why

E. Understands how and when to use standard differentiation and integration techniques

- Uses standard differentiation techniques
- Uses standard integration techniques
- Understands the relationship between position, velocity, and acceleration functions of a particle in motion

F. Understands how to analyze the behavior of a function; e.g., extrema, concavity, symmetry

- Uses the first and second derivatives to analyze the graph of a function

G. Understands how to apply derivatives to solve problems; e.g., related rates, optimization

- Applies derivatives to solve problems

H. Understands the foundational theorems of calculus; e.g., fundamental theorems of calculus, mean value theorem, intermediate value theorem

- Solves problems using the foundational theorems of calculus
- Understands the relationship between differentiation and integration, including the role of the fundamental theorems of calculus
- Matches graphs of functions with graphs of their derivatives or accumulations
- Understands how to use differentiation and integration of a function to express rates of change and total change
- Understands and calculates the average value of a function over an interval; i.e., mean value theorem of integrals

I. Understands how to use integration to compute area, volume, distance, or other accumulation processes

- Uses integration techniques to compute area, volume, distance, or other accumulation processes

J. Knows how to determine the limits of sequences, if they exist

- Determines the limits of sequences when they exist

K. Is familiar with simple infinite series

- Determines if simple infinite series converge or diverge
- Finds the sum of a simple infinite series if it exists
- Finds the partial sum of a simple infinite series
- Models phenomena (e.g., compound interest, annuities, growth, decay) using finite and infinite arithmetic and geometric sequences and series

Test II Subareas

Subarea	Approx. Percentage of Test
I. Functions	40%
II. Geometry	30%
III. Probability and Statistics	30%

Test II Objectives

Subarea I: Functions

Objective 1: Understands how to interpret functions and apply knowledge to build functions

The beginning Mathematics teacher:

A. Understands the function concept and the use of function notation

- Understands that a function from one set (called the domain) to another set (called the range) assigns to each element of the domain exactly one element of the range
- Uses function notation, evaluates functions, and interprets statements that use function notation in terms of a context
- Recognizes that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers
- Determines the domain and range of a function from a function rule (e.g., $f(x) = 2x + 1$), graph, set of ordered pairs, or table

B. Understands how function behavior is analyzed using different representations; e.g., graphs, mappings, tables

- For a function that models a relationship between two quantities, interprets key features of graphs and tables (e.g., increasing/decreasing, maximum/minimum, periodicity) in terms of the quantities
- Given a verbal description of a relation, sketches graphs that show key features of that relation
- Graphs functions (i.e., radical, piecewise, absolute value, polynomial, rational, logarithmic, trigonometric) expressed symbolically, and identifies key features of the graph
- Writes a function that is defined by an expression in different but equivalent forms to reveal different properties of the function; e.g., zeros, extreme values, symmetry of the graph

- Interprets the behavior of exponential functions; e.g., growth, decay
- Understands how to determine if a function is odd, even, or neither, and any resulting symmetries

C. Understands how functions and relations are used to model relationships between quantities

- Writes a function that relates two quantities
- Determines an explicit expression or a recursive process that builds a function from a context

D. Understands how new functions are obtained from existing functions; e.g., compositions, transformations, inverses

- Describes how the graph of $g(x)$ is related to the graph of $f(x)$, where $g(x) = f(x) + k$, $g(x) = k\,f(x)$, $g(x) = f(kx)$, or $g(x) = f(x + k)$ for specific values of k (both positive and negative), and finds the value of k given the graphs
- Determines if a function has an inverse and writes an expression for the inverse
- Verifies by composition if one function is the inverse of another
- Given that a function f has an inverse, finds values of the inverse function from a graph or a table of f
- Given a noninvertible function, determines the largest possible domain of the function that produces an invertible function
- Understands the inverse relationship between exponential and logarithmic functions, and uses this relationship to solve problems
- Combines standard function types using arithmetic operations
- Performs domain analysis on functions resulting from arithmetic operations
- Composes functions algebraically, numerically, and graphically
- Performs domain analysis on functions resulting from compositions

Objective 2: Understands and applies knowledge of linear, quadratic, and exponential models and trigonometric functions

The beginning Mathematics teacher:

A. Understands differences between linear, quadratic, and exponential models, including how their equations are created and used to solve problems

- Understands that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals
- Recognizes situations in which one quantity changes at a constant rate per unit interval relative to another
- Recognizes situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another

- Constructs linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two ordered pairs (including reading these from a table)
- Observes that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function
- Expresses the solution to an exponential equation with base *b* as a logarithm; e.g., $3 \cdot 2^{5t} = 20$, $3 \cdot e^{5t} = 20$
- Uses technology to evaluate logarithms that have any base
- Interprets the parameters in a linear or exponential function in terms of a context; e.g., $A(t) = Pe^{rt}$
- Uses quantities that are inversely related to model phenomena

B. Understands how to construct the unit circle and how to use it to find values of trigonometric functions for all angle measures in their domains

- Finds the values of trigonometric functions of any angle
- Uses the unit circle to explain symmetry and periodicity of trigonometric functions

C. Understands how periodic phenomena are modeled using trigonometric functions

- Chooses trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline
- Understands how to restrict the domain of a trigonometric function so that its inverse can be constructed
- Uses inverse functions to solve trigonometric equations that arise in modeling contexts, and interprets them in terms of the context

D. Understands the application of trigonometric identities (e.g., Pythagorean, double angle, half angle, sum of angles, difference of angles)

- Proves Pythagorean identities (e.g., $\sin^2 \theta + \cos^2 \theta = 1$) and uses them to solve problems
- Uses trigonometric identities to rewrite expressions and solve equations
- Understands trigonometric identities in the context of equivalent graphs of trigonometric functions; e.g., $y = \sin x$ and $y = \cos\left(\frac{\pi}{2} - x\right)$ are equivalent graphs

E. Knows how to interpret representations of functions of two variables; e.g., three-dimensional graphs, tables

- Interprets representations of functions of two variables; e.g., $z = f(x, y)$

F. Understands how to solve trigonometric, logarithmic, and exponential equations

- Solves trigonometric, logarithmic, and exponential equations

Subarea II: Geometry

Objective 1: Understands congruence/similarity/triangles/trigonometric ratios and equations for geometric properties

The beginning Mathematics teacher:

A. Understands transformations in a plane

- Knows precise definitions of angle, circle, line segment, perpendicular lines, and parallel lines
- Represents transformations in the plane
- Recognizes whether a transformation preserves distance and angle measure
- Given a rectangle, parallelogram, trapezoid, or regular polygon, describes the rotations and reflections that map it onto itself
- Develops definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments
- Given a geometric figure and a rotation, reflection, or translation, draws the transformed figure
- Specifies a sequence of transformations that will map a given figure onto another figure

B. Understands how to prove geometric theorems such as those about lines and angles, triangles, and parallelograms

- Proves theorems about lines and angles
- Proves theorems about triangles
- Proves theorems about parallelograms

C. Understands how geometric constructions are made with a variety of tools and methods

- Recognizes formal geometric constructions
- Explains how formal geometric constructions are made; e.g., an equilateral triangle, a square, a regular hexagon inscribed in a circle

D. Understands congruence and similarity in terms of transformations

- Uses geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure
- Verifies the properties of dilations given by a center and a scale factor
- Given two figures, uses the definition of congruence in terms of rigid motions to decide if they are congruent
- Given two figures, uses the definition of similarity in terms of dilations to decide if they are similar
- Explains how the criteria for triangle congruence (e.g., ASA, SAS, SSS, HL) follow from the definition of congruence in terms of rigid motions

- Uses the properties of similarity transformations to establish the AA criterion for two triangles to be similar
- Uses congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures

E. Understands how trigonometric ratios are defined in right triangles

- Understands that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles
- Explains and uses the relationship between the sine and cosine of complementary angles
- Uses trigonometric ratios and the Pythagorean theorem to solve right triangles in applied problems

F. Understands how trigonometry is applied to general triangles

- Uses the formula $A = \frac{1}{2}ab\sin(C)$ for the area of a triangle to solve problems
- Applies the Law of Sines and the Law of Cosines to find unknown measurements in triangles

G. Knows how to translate between a geometric description (e.g., focus, asymptotes, directrix) and an equation for a conic section

- Determines and uses the equation of a circle of given center and radius
- Finds the center and radius of a circle given by an equation in standard form
- Determines the equation of a parabola given a focus and directrix
- Determines and uses the equations of ellipses and hyperbolas given the foci, using the fact that the sum or difference of distances from a point on the curve to the foci is constant

H. Understands how to use coordinate geometry to algebraically prove simple geometric theorems

- Uses coordinates to prove simple geometric theorems algebraically
- Proves the slope criteria for parallel and perpendicular lines, and uses parallel and perpendicular lines to solve geometric problems
- Finds the point on a directed line segment between two given points that partitions the segment in a given ratio
- Uses coordinates to compute perimeters of polygons and areas of triangles and quadrilaterals

Objective 2: Understands circles, geometric measurement and dimension, and modeling with geometry

The beginning Mathematics teacher:

A. Understands and applies theorems about circles
    - Identifies and describes relationships among inscribed angles, radii, and chords
    - Proves properties of angles for a quadrilateral inscribed in a circle
    - Constructs a tangent line from a point outside a given circle to the circle
B. Understands arc length and area measurements of sectors of circles
    - Uses the length of the arc intercepted by a central angle or inscribed angle to solve circumference problems
    - Uses the formula for the area of a sector to solve problems
C. Understands how perimeter, area, surface area, and volume formulas are used to solve problems
    - Uses the perimeter and area of geometric shapes to solve problems
    - Uses the surface area and volume of prisms, cylinders, pyramids, cones, and spheres to solve problems
D. Knows how to visualize relationships (e.g., cross section, nets, rotations) between two-dimensional and three-dimensional objects
    - Identifies the shapes of two-dimensional cross sections of three-dimensional objects, and identifies three-dimensional objects generated by rotations of two-dimensional objects
    - Uses two-dimensional representations of three-dimensional objects to visualize and solve problems
E. Knows how to apply geometric concepts in real-world situations
    - Uses geometric shapes, their measures, and their properties to describe objects
    - Applies geometric methods to solve design problems
F. Understands the properties of parallel and perpendicular lines, triangles, quadrilaterals, polygons, and circles and how they can be used in problem solving
    - Solves problems involving parallel, perpendicular, and intersecting lines
    - Applies angle relationships (e.g., supplementary, vertical, alternate interior) to solve problems
    - Solves problems that involve medians, midpoints, and altitudes
    - Solves problems involving special triangles; e.g., isosceles, equilateral, right
    - Knows geometric properties of and relationships among quadrilaterals; e.g., parallelograms, trapezoids

- Solves problems involving angles and diagonals
- Solves problems involving polygons with more than four sides

Subarea III: Probability and Statistics

Objective 1: Understands how to interpret categorical and quantitative data, make inferences, and justify conclusions

The beginning Mathematics teacher:

A. Understands how to summarize, represent, and interpret data collected from measurements on a single variable; e.g., box plots, dot plots, normal distributions

   - Represents data with plots on the real number line; e.g., dot plots, histograms, and box plots
   - Uses statistics appropriate to the shape of the data distribution to compare center (e.g., median, mean) and spread (e.g., interquartile range, standard deviation) of two or more different data sets
   - Interprets differences in shape, center, and spread in the context of the data sets, accounting for possible effects of outliers
   - Uses the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages, and recognizes that there are data sets for which such a procedure is not appropriate

B. Understands how to summarize, represent, and interpret data collected from measurements on two variables, either categorical or quantitative; e.g., scatterplots, time series

   - Summarizes and interprets categorical data for two categories in two-way frequency tables; e.g., joint, marginal, conditional relative frequencies
   - Recognizes possible associations and trends in the data
   - Represents data for two quantitative variables on a scatterplot, and describes how the variables are related

C. Understands how to create and interpret linear regression models; e.g., rate of change, intercepts, correlation coefficient

   - Uses technology to fit a function to data (i.e., linear regression) and determines a linear correlation coefficient
   - Uses functions fitted to data to solve problems in the context of the data
   - Assesses the fit of a function by plotting and analyzing residuals
   - Interprets the slope and the intercept of a regression line in the context of the data
   - Interprets a linear correlation coefficient
   - Distinguishes between correlation and causation

D. Understands statistical processes and how to evaluate them

- Understands statistics as a process for making inferences about population parameters based on a random sample from that population
- Decides if a specified model is consistent with results from a given data-generating process; e.g., using simulation

E. Understands how to make inferences and justify conclusions from samples, experiments, and observational studies

- Recognizes the purposes of and differences among sample surveys, experiments, and observational studies, and explains how randomization relates to each
- Uses data from a sample survey to estimate a population mean or proportion
- Uses data from a randomized experiment to compare two treatments
- Uses results of simulations to decide if differences between parameters are significant
- Evaluates reports based on data

Objective 2: Understands conditional probability, the rules of probability, and using probability to make decisions

The beginning Mathematics teacher:

A. Understands the concepts of independence and conditional probability and how to apply these concepts to data

- Describes events as subsets of a sample space using characteristics of the outcomes, or as unions, intersections, or complements of other events
- Understands that two events, *A* and *B*, are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$
- Understands the conditional probability of *A* given *B* as $\frac{P(A \text{ and } B)}{P(B)}$, and interprets independence of *A* and *B* as saying that $P(A|B) = P(A)$ and $P(B|A) = P(B)$

B. Understands how to compute probabilities of simple events, probabilities of compound events, and conditional probabilities

- Calculates probabilities of simple and compound events
- Constructs and interprets two-way frequency tables of data when two categories are associated with each object being classified; uses the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities
- Finds $P(A|B)$, and interprets it in terms of a given model

- Applies the addition rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interprets it in terms of a given model
- Applies the general multiplication rule in a uniform probability model, $P(A \text{ and } B) = P(A)P(B|A) = P(B)P(A|B)$, and interprets it in terms of a given model
- Calculates probabilities using the binomial probability distribution

C. Knows how to make informed decisions using probabilities and expected values

- Defines a random variable for a quantity of interest by assigning a numerical value to each event in a sample space, and graphs the corresponding probability distribution using the same graphical displays as for data distributions
- Calculates the expected value of a random variable, and interprets it as the mean of the probability distribution
- Develops a probability distribution for a random variable, defined for a sample space in which theoretical probabilities can be calculated, and finds the expected value
- Develops a probability distribution for a random variable, defined for a sample space in which probabilities are assigned empirically, and finds the expected value
- Weighs the possible outcomes of a decision by assigning probabilities to payoff values and finding expected values
- Analyzes decisions and strategies using probability concepts; e.g., fairness

D. Understands how to use simulations to construct experimental probability distributions and to make informal inferences about theoretical probability distributions

- Given the results of simulations, constructs experimental probability distributions
- Given the results of simulations, makes informal inferences about theoretical probability distributions

E. Understands how to find probabilities involving finite sample spaces and independent trials

- Uses the fundamental counting principle to find probabilities involving finite sample spaces and independent trials

Practice Questions

The practice questions in this study companion are designed to familiarize you with the types of questions you may see on the assessment. While they illustrate some of the formats and types of questions you will see on the test, your performance on these sample questions should not be viewed as a predictor of your performance on the actual test. Fundamentally, the most important component in ensuring your success is familiarity with the content that is covered on the assessment.

To respond to a practice question, choose one of the answer options listed. Be sure to read the directions carefully to ensure that you know what is required for each question. You may find it helpful to time yourself to simulate actual testing conditions. A correct answer and a rationale for each sample test question are in the section following the practice questions.

Keep in mind that the test you take at an actual administration will have different questions, although the proportion of questions in each subarea will be approximately the same. You should not expect the percentage of questions you answer correctly in these practice questions to be exactly the same as when you take the test at an actual administration, since numerous factors affect a person's performance in any given testing situation.

This test has Notations, Definitions, and Formulas built into the testing software. This reference document can be accessed by selecting the "Help" function. The test clock does not stop when the "Help" function is being used.

NOTATIONS

(a, b)	$\{x: a < x < b\}$
$[a, b)$	$\{x: a \leq x < b\}$
$(a, b]$	$\{x: a < x \leq b\}$
$[a, b]$	$\{x: a \leq x \leq b\}$

gcd (m,n)	<u>greatest common divisor</u> of two integers m and n
lcm (m, n)	<u>least common multiple</u> of two integers m and n
$[x]$	<u>greatest integer</u> m such that $m \leq x$
$m \equiv k \pmod{n}$	m and k are <u>congruent modulo n</u> (m and k have the same remainder when divided by n, or equivalently, $m - k$ is a multiple of n)
f^{-1}	<u>inverse</u> of an invertible function f; (<u>not</u> to be read as $\frac{1}{f}$)
$\lim_{x \to a^+} f(x)$	<u>right-hand limit</u> of $f(x)$; limit (if it exists) of $f(x)$ as x approaches a from the right
$\lim_{x \to a^-} f(x)$	<u>left-hand limit</u> of $f(x)$; limit (if it exists) of $f(x)$ as x approaches a from the left
$\varnothing$	the empty set
$x \in S$	x is an element of set S
$S \subset T$	set S is a proper subset of set T
$S \subseteq T$	either set S is a proper subset of set T or $S = T$
$\overline{S}$	complement of set S; the set of all elements not in S that are in some specified universal set
$T \setminus S$	relative complement of set S in set T; i.e., the set of all elements of T that are not elements of S
$S \cup T$	union of sets S and T
$S \cap T$	intersection of sets S and T

DEFINITIONS

Discrete Mathematics

A relation $\Re$ on a set S is

> reflexive if $x \Re x$ for all $x \in S$
> symmetric if $x \Re y \Rightarrow y \Re x$ for all $x, y \in S$
> transitive if $(x \Re y \text{ and } y \Re z) \Rightarrow x \Re z$ for all $x, y, z \in S$
> antisymmetric if $(x \Re y \text{ and } y \Re x) \Rightarrow x = y$ for all $x, y \in S$

An equivalence relation is a reflexive, symmetric, and transitive relation.

FORMULAS

Sum

$\sin(x + y) = \sin x \cos y + \cos x \sin y$
$\cos(x + y) = \cos x \cos y - \sin x \sin y$

Half-Angle (sign depends on the quadrant of $\frac{\theta}{2}$)

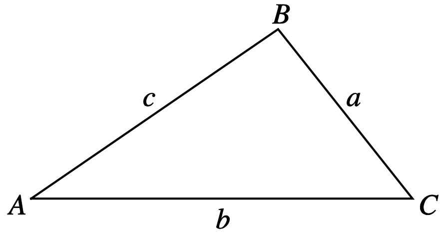
$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$
$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

Range of inverse trigonometric functions

$\sin^{-1} x$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[0, \pi]$
$\tan^{-1} x$	$(-\pi/2, \pi/2)$

Law of Sines

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$



Law of Cosines

$c^2 = a^2 + b^2 - 2ab(\cos C)$

DeMoivre's Theorem

$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

Coordinate Transformation

Rectangular (x, y) to polar (r, θ): $r^2 = x^2 + y^2$; $\tan\theta = \frac{y}{x}$ if $x \neq 0$

Polar (r, θ) to rectangular (x, y): $x = r\cos\theta$; $y = r\sin\theta$

Distance from point (x_1, y_1) to line $Ax + By + C = 0$

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

Volume

Sphere: radius r	$V = \frac{4}{3}\pi r^3$
Right circular cone: height h, base of radius r	$V = \frac{1}{3}\pi r^2 h$
Right circular cylinder: height h, base of radius r	$V = \pi r^2 h$
Pyramid: height h, base of area B	$V = \frac{1}{3}Bh$
Right prism: height h, base of area B	$V = Bh$

Surface Area

Sphere: radius r $\qquad A = 4\pi r^2$

Lateral surface area of
right circular cone: radius r, slant height s $\qquad A = \pi rs$

Differentiation

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$
$$(f(g(x)))' = f'(g(x))g'(x)$$
$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2} \text{ if } g(x) \neq 0$$

Integration by Parts

$$\int u\, dv = uv - \int v\, du$$

Directions: Each of the questions or incomplete statements below is followed by four suggested answers or completions. Select the one that is best in each case.

1. The orthogonal projection of 3-space onto the *xy*-plane takes the point (x, y, z) onto the point $(x, y, 0)$. This transformation can be represented by the matrix equation $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$, where M is which of the following matrices?

A. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

B. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

C. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

D. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Answer and Rationale

2. For what value of x is the matrix $\begin{pmatrix} 1 & 4 \\ x & 6 \end{pmatrix}$ NOT invertible?

A. $-\frac{3}{2}$

B. 0

C. $\frac{3}{2}$

D. 2

Answer and Rationale

3. If x and y are even numbers and $z = 2x^2 + 4y^2$, then the greatest even number that must be a divisor of z is

A. 2

B. 4

C. 8

D. 16

Answer and Rationale

4. What is the units digit of 33^{408} ?

A. 1

B. 3

C. 7

D. 9

Answer and Rationale

5. $$\frac{L}{W}=\frac{W}{L-W}$$

Ancient Greeks thought that a rectangle was most aesthetically pleasing to the eye when its length L and width W satisfied the golden-ratio equation shown above. If a rectangle with width 40 satisfies the golden-ratio equation, what is the length of the rectangle?

A. $20-20\sqrt{5}$

B. $20+20\sqrt{5}$

C. $40-40\sqrt{5}$

D. $40+40\sqrt{5}$

Answer and Rationale

6. Which of the following is the set of all real values of x that satisfy the inequality $3+x-x^2>|2x-3|$?

A. $0<x<2$

B. $1<x<3$

C. $x<0$ or $x>2$

D. $x<1$ or $x>3$

Answer and Rationale

7. A taxicab driver charges a fare of \$2.00 for the first quarter mile or less and \$0.75 for each quarter mile after that. Which of the following equations models the fare, f, in dollars, for a ride m miles long, where m is a positive integer?

A. $f = 2.00 + 0.75(m - 1)$

B. $f = 2.00 + 0.75\left(\frac{m}{4} - 1\right)$

C. $f = 2.00 + 0.75(4m - 1)$

D. $f = 2.00 + 0.75(4(m - 1))$

Answer and Rationale

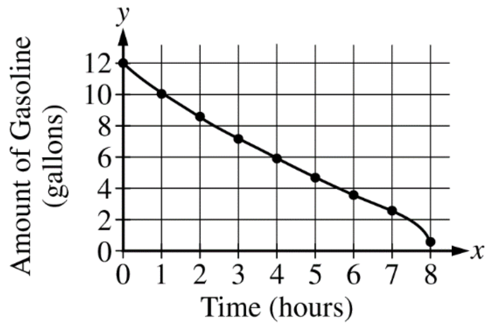
8. $$6x + 3y = 24$$

What are the coordinates of the x- and y-intercepts of the graph of the equation above in the xy-plane?

A. $(8,0)$ and $(0,4)$
B. $(6,0)$ and $(0,3)$
C. $(4,0)$ and $(0,8)$
D. $(4,0)$ and $(0,3)$

Answer and Rationale

GASOLINE IN JEANNE'S CAR



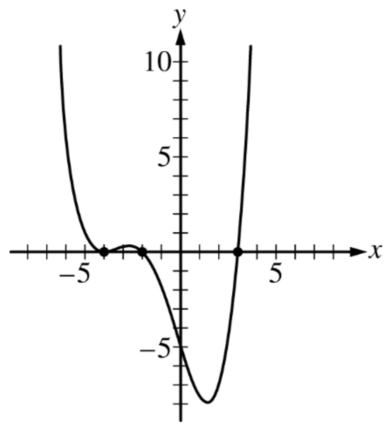
9. The curve in the xy-plane above represents the amount of gasoline, y, in gallons, in Jeanne's car for x hours, where $0 \le x \le 8$. Which of the following is closest to the average rate of change, in gallons per hour, of the gasoline in Jeanne's car over the interval $1 \le x \le 6$?

A. −0.67
B. −0.83
C. −1.20
D. −1.50

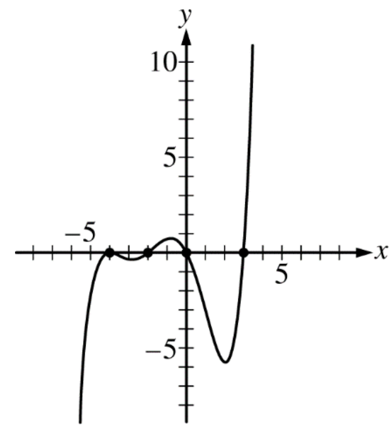
Answer and Rationale

10. The function f is a fourth-degree polynomial function with real coefficients, and $f(-4) = f(-2) = f(3) = 0$ with no other zeros. Which of the following could be a graph of $y = f(x)$ in the xy-plane?

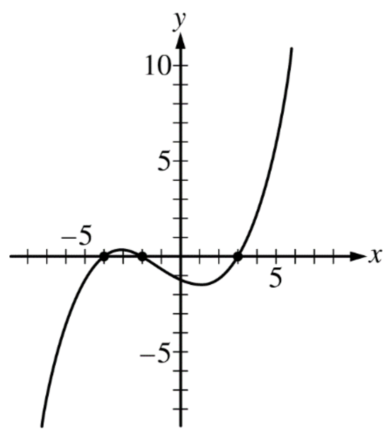
A.



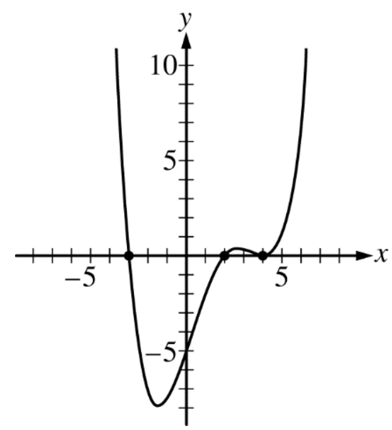
B.



C.



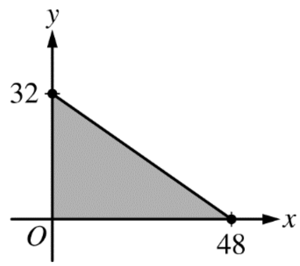
D.



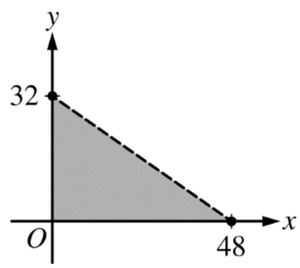
Answer and Rationale

11. Melissa is selling jewelry to raise money for the student council. She sells each bracelet for \$8 and each necklace for \$12. Melissa's goal is to raise at least \$384 for the student council. If x represents the number of bracelets that Melissa sells and y represents the number of necklaces that Melissa sells, which of the following graphs represents the number of bracelets and necklaces Melissa can sell to meet her goal?

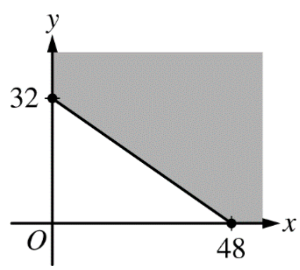
A.



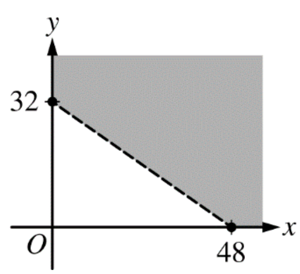
B.



C.



D.



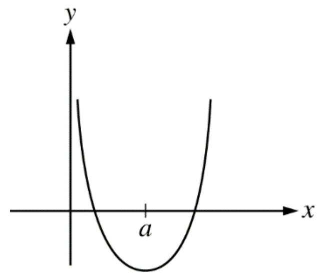
Answer and Rationale

12. **For the following question, enter your answer in the answer box.**

Let function f be defined for all positive integers by $f(1) = 2$ and by the recursion $f(k) = 2f(k-1) - 4$ for all integers $k \geq 2$. What is the value of $f(5)$?

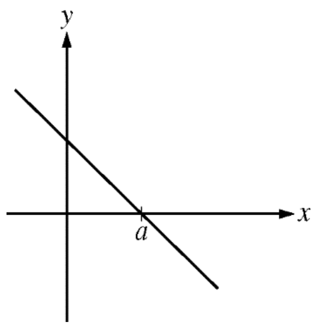
$f(5) =$ $\boxed{}$

Answer and Rationale

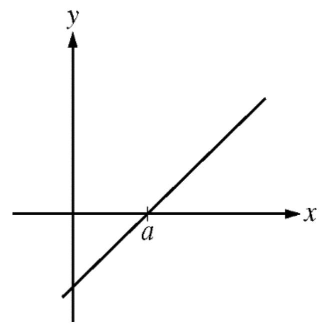


13. The figure above is a graph of a differentiable function f. Which graph could be the graph of the first derivative of this function?

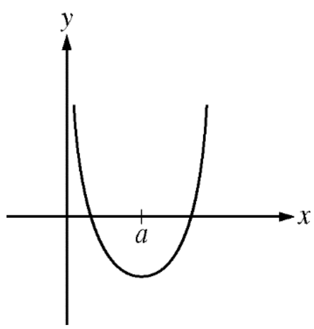
A.



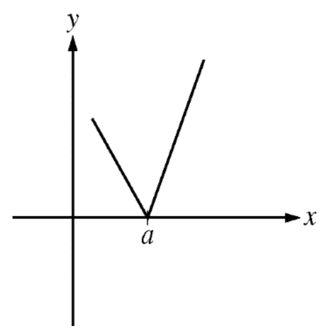
B



C



D.

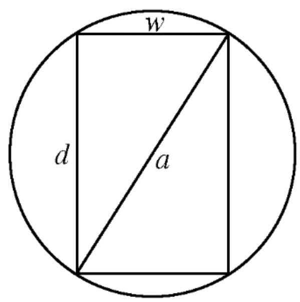


Answer and Rationale

14. In a certain chemical reaction, the number of grams, N, of a substance produced t hours after the reaction begins is given by $N(t) = 16t - 4t^2$, where $0 < t < 2$. At what instantaneous rate, in grams per hour, is the substance being produced 30 minutes after the reaction begins?

A. 7
B. 12
C. 16
D. 20

Answer and Rationale



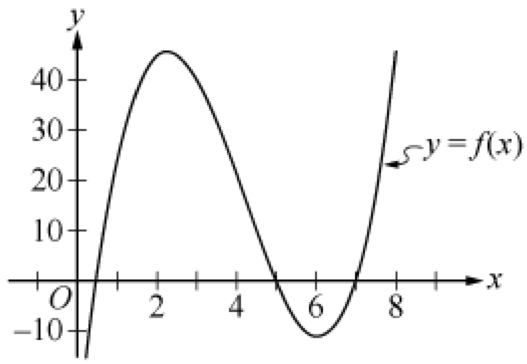
15. A rectangular beam is to be cut from a log with a circular cross section of diameter a, as shown in the figure above. The strength of the beam, S, can be modeled by the equation

$$S(w) = kw\left(a^2 - w^2\right),$$

where k is a positive constant. Which of the following shows the relationship between the dimensions d and w of the beam having maximum strength?

A. $d = \frac{w}{2}$
B. $d = \frac{w}{\sqrt{2}}$
C. $d = \sqrt{2}w$
D. $d = 2w$

Answer and Rationale



16. The xy-plane above shows the graph of the function $f(x) = 2x^3 - 25x^2 + 82x - 35$ defined for all real numbers x. What happens to the slope of the secant line that passes through the points $(x, f(x))$ and $(2, f(2))$ as x approaches 2 ?

A. The slope approaches 0.
B. The slope approaches 6.
C. The slope approaches 28.
D. The slope approaches 55.

Answer and Rationale

17. If $f(x) = 3x^2$, what are all real values of a and b for which the graph of $g(x) = ax^2 + b$ is below the graph of $f(x)$ for all values of x ?

A. $a \geq 3$ and b is positive.
B. $a \leq 3$ and b is negative.
C. a is negative and b is positive.
D. a is any real number and b is negative.

Answer and Rationale

$$P(t) = 250 \cdot (3.04)^{\frac{t}{1.98}}$$

18. At the beginning of 2010, the population of rabbits in a wooded area was 250. The function above was used to model the approximate population, P, of rabbits in the area t years after January 1, 2010. According to this model, which of the following best describes how the rabbit population changed in the area?

    A. The rabbit population doubled every 4 months.
    B. The rabbit population tripled every 6 months.
    C. The rabbit population doubled every 36 months.
    D. The rabbit population tripled every 24 months.

Answer and Rationale

19. In the xy-plane, the graphs of functions $f(x) = -x$ and $g(x) = e^x$ intersect at point A, which is also on the graph of function $h(x) = bx - 1$. Which of the following is closest to the value of b ?

    A. −2.72
    B. −2.76
    C. −2.80
    D. −2.84

Answer and Rationale

$$f(x) = 2x + 4$$

$$g(x) = \frac{x}{2} + b$$

20. Functions f and g are defined by the equations above for all numbers x, where b is a constant. If $(f \circ g)(x) = (g \circ f)(x)$ for all values of x, what is the value of b ?

    A. -4
    B. -2
    C. 2
    D. 4

Answer and Rationale

21. Let f be the function defined by $f(x) = \sqrt{x^2 - 4}$ for all real numbers x for which $f(x)$ is a real number. What is the domain of function f ?

    A. $[2, \infty)$
    B. $[4, \infty)$
    C. $(-\infty, -2] \cup [2, \infty)$
    D. $(-\infty, -4] \cup [4, \infty)$

Answer and Rationale

22. A group of 5 students plan to buy a present that costs n dollars for their teacher, and they initially plan to share the cost equally. However, 2 of the students later decide not to share the cost of the present. The cost per student when only 3 students share the cost equally is d dollars greater than the cost per student when all 5 students share the cost equally. Which of the following expresses n as a function of d ?

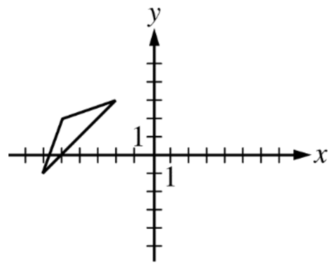
A. $n = \frac{d}{7.5}$

B. $n = \frac{d}{15}$

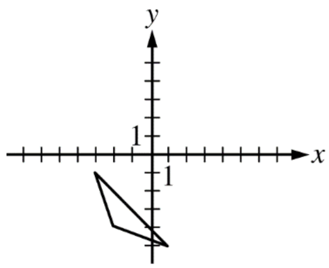
C. $n = 7.5d$

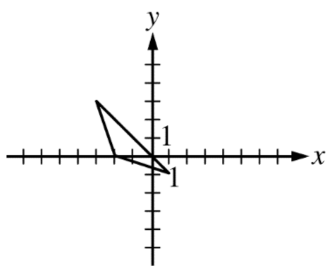
D. $n = 15d$

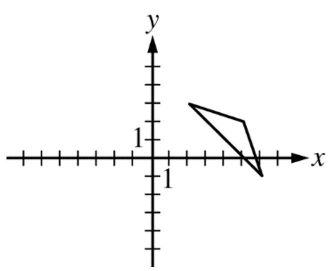
Answer and Rationale

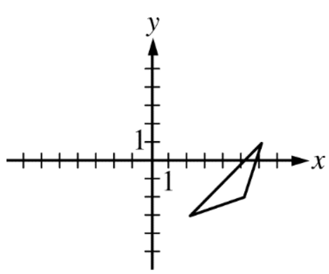


23. The triangle in the *xy*-plane above is translated 3 units right and 2 units up, and then it is reflected across the *x*-axis. Which of the following is the result of this transformation?

A. 

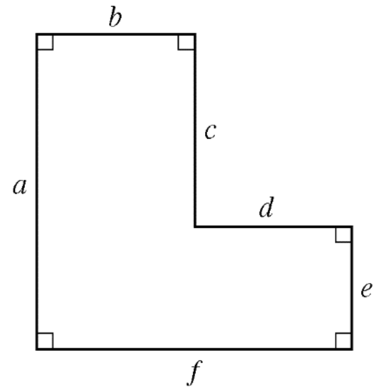
B. 

C. 

D. 

Answer and Rationale

For the following question, select all the answer choices that apply.



24. If the perimeter and the area of the region above are P and A, respectively, which of the following statements are true?

    Select all that apply.

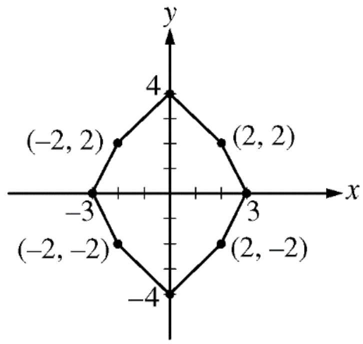
    A. $f = b + d$
    B. $P = 2a + 2f$
    C. $A = af - cd$

Answer and Rationale

25. In ΔABC (not shown), the length of side $\overline{AB}$ is 12, the length of side $\overline{BC}$ is 9, and the measure of angle BAC is 30°. What is the length of side $\overline{AC}$?

    A. 17.10
    B. 4.73
    C. 3.68
    D. It cannot be determined from the information given.

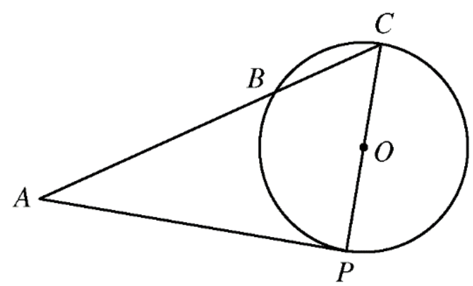
Answer and Rationale



26. For how many angles θ, where $0 < \theta \leq 2\pi$, will rotation about the origin by angle θ map the octagon in the figure above onto itself?

    A. One
    B. Two
    C. Four
    D. Eight

Answer and Rationale



Note: Figure not drawn to scale.

27. In the circle above with center O and radius 2, tangent $\overline{AP}$ has length 3 and is tangent to the circle at P. If $\overline{CP}$ is a diameter of the circle, what is the length of $\overline{BC}$?

    A. 1.25
    B. 2
    C. 3.2
    D. 5

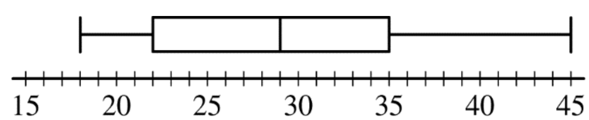
Answer and Rationale

Stem	Leaf
9	1 3 4 5 7
8	2 2 5 6 6 8 9
7	0 2 4 5 8 8
6	1 3 7 9

28. The stem-and-leaf plot above shows the course grades that each of 22 students received in a history course. The course grade is represented by using the tens digit of each grade as a stem and the corresponding units digit as a leaf. For example, the stem 9 and the leaf 1 in the first row of the table represent a grade of 91. What is the median course grade of the 22 students?

    A. 78
    B. 80
    C. 80.7
    D. 82

Answer and Rationale



29. The boxplot above summarizes the times, in minutes, that it took 20 people to finish a race. Based on the boxplot, which of the following is a reasonable conclusion?

    A. The mean of the times for all 20 people was 29 minutes.
    B. The number of runners who finished the race in 35 to 45 minutes is greater than the number of runners who finished in 18 to 22 minutes.
    C. More than half the runners finished the race in 22 to 35 minutes.
    D. At least one person finished in 45 minutes.

Answer and Rationale

Test	Martha's Score	Mean	Standard Deviation
English	74	76	5.0
Math	73	63	6.0
History	63	58	5.5
Biology	69	60	4.5

30. The table above shows Martha's score on each of 4 tests and the mean and standard deviation of the test scores for her class on each of the tests. On which test did Martha have the highest standardized score?

A. English
B. Math
C. History
D. Biology

Answer and Rationale

31. The measures of the handspans of ninth-grade students at Tyler High School are approximately normally distributed, with a mean of 7 inches and a standard deviation of 1 inch. Of the following, which group is expected to have the greatest percent of measures?

A. The group of handspan measures that are less than 6 inches
B. The group of handspan measures that are greater than 7 inches
C. The group of handspan measures that are between 6 and 8 inches
D. The group of handspan measures that are between 5 and 7 inches

Answer and Rationale

32. A two-sided coin is unfairly weighted so that when it is tossed, the probability that heads will result is twice the probability that tails will result. If the coin is to be tossed 3 separate times, what is the probability that tails will result on exactly 2 of the tosses?

A. $\frac{2}{9}$

B. $\frac{3}{8}$

C. $\frac{4}{9}$

D. $\frac{2}{3}$

Answer and Rationale

Answer Key and Rationales

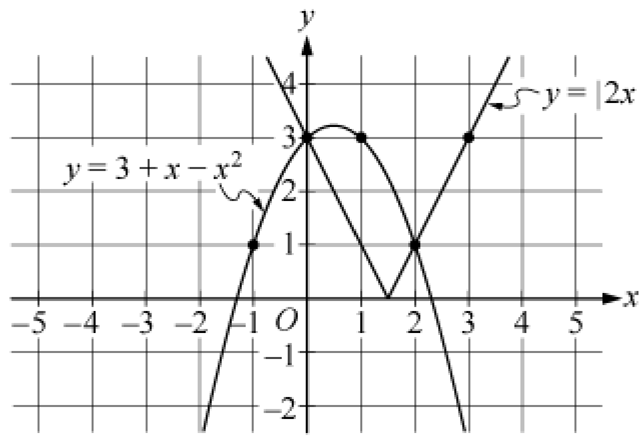
Question Number	Correct Answer	Rationale
1	B	**Option B is correct.** In order to answer this question, consider how matrix multiplication is performed. The problem asks to find a matrix, M, that when multiplied by any matrix of the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, yields the result $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$. Notice that all of the options are 3×3 matrices. This problem can be solved for the general case or reasoned to the answer. First, the general solution: Let $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$. Then $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax+by+cz \\ dx+ey+fz \\ gx+hy+jz \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Since $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ for all $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then $\begin{cases} ax+by+cz = x \\ dx+ey+fz = y \\ gx+hy+jz = 0 \end{cases}$ for all x, y, and z. This implies $a = 1$, $b = 0$, $c = 0$; and $d = 0$, $e = 1$, $f = 0$; and $g = h = j = 0$; and, therefore, $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The correct answer is option B. One could also determine the answer by inspecting the options given. Since multiplying the first row of M by the matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ has to result in only the x term for all x, y, and z, the first entry in the

Question Number	Correct Answer	Rationale
		first row must be 1 and the other 0. Likewise, multiplying the second row of M by the matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ will result only in the y term for all x, y, and z, so the entries in the second row must all be 0, 1, 0, in that order. Multiplying the third row of M by the matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ results in 0 for all x, y, and z, so the entries in the third row must all be 0. Therefore, $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and the correct answer is option B. ***Back to Question***
2	C	**Option C is correct.** A matrix is not invertible if the determinant of the matrix is equal to zero. The determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is equal to $ad - bc$. For the matrix given in the question, the determinant is equal to $(1)(6) - (4)(x)$. This equals 0 when $6 - 4x = 0$, or $x = \frac{3}{2}$, and the correct answer is option C. ***Back to Question***

Question Number	Correct Answer	Rationale
3	C	**Option C is correct.** Since 2 is a divisor of both $2x^2$ and $4y^2$, it follows that 2 is a divisor of z. To find out if there is a greater even number that must be a divisor of z, consider the additional information given, which is that x and y are both even numbers. Since x and y are even numbers, they can be expressed as $x = 2m$ and $y = 2n$, respectively, where m and n can be either odd or even integers. Substituting these values for x and y into the expression for z yields $z = 2(2m)^2 + 4(2n)^2$. It follows then that $z = 8m^2 + 16n^2$ and that 8 is a divisor of z. The number 16 would also be a divisor of z if m is even, but not if m is odd. Since m and n can be either even or odd and the question asks for the largest even number that must be a divisor of z, the correct answer is 8, option C. ***Back to Question***
4	A	**Option A is correct.** To find the units digit of 33^{408}, it is helpful to find the first few integer powers of 33 and look for a pattern. For example, $33^1 = 33$ $33^2 = 1{,}089$ $33^3 = 35{,}937$ $33^4 = 1{,}185{,}921$ $33^5 = 39{,}135{,}393$ $33^6 = 1{,}291{,}467{,}969$ The pattern in the units digit is 3, 9, 7, 1, 3, 9, ... and it will continue to repeat with every four integers of the exponent. Dividing 408 by 4 yields 102 with no remainder. Therefore, the units digit of 33^{408} will be the same as the units digit of 33^4 , which is 1. So, the correct answer is option A. ***Back to Question***

Question Number	Correct Answer	Rationale
5	B	**Option B is correct.** After cross multiplying, the golden-ratio equation $\frac{L}{W} = \frac{W}{L-W}$ becomes $L(L-W) = W^2$, or equivalently $L^2 - LW - W^2 = 0$. After substituting $W = 40$, the equation becomes $L^2 - 40L - 1{,}600 = 0$, which can be solved using the quadratic formula, as shown below. $$\begin{aligned} L &= \frac{-(-40) \pm \sqrt{(-40)^2 - 4(1)(-1{,}600)}}{2(1)} \\ &= \frac{40 \pm \sqrt{1{,}600 + 4(1{,}600)}}{2} \\ &= \frac{40 \pm \sqrt{1{,}600(5)}}{2} \\ &= \frac{40 \pm \sqrt{8{,}000}}{2} \\ &= \frac{40 \pm 40\sqrt{5}}{2} \\ &= 20 \pm 20\sqrt{5} \end{aligned}$$ We can rule out $L = 20 - 20\sqrt{5}$ because $20 - 20\sqrt{5}$ is negative, so it cannot be the length of a rectangle. Therefore, $L = 20 + 20\sqrt{5}$, and the correct answer is option B. ***Back to Question***

Question Number	Correct Answer	Rationale
6	A	**Option A is correct.** The inequality in the problem is equivalent to $\lvert 2x-3\rvert < -x^2+x+3$, which is equivalent to the compound inequalities $-\left(-x^2+x+3\right) < 2x-3$ and $2x-3 < -x^2+x+3$. The first of the two compound inequalities is equivalent to $x\left(x-3\right) < 0$, as shown below. The solution set of the inequality $-\left(-x^2+x+3\right) < 2x-3$ is the interval $(0,3)$. $\begin{aligned} -\left(-x^2+x+3\right) &< 2x-3 \\ x^2-x-3 &< 2x-3 \\ x^2-x-3-2x+3 &< 0 \\ x^2-3x &< 0 \\ x\left(x-3\right) &< 0 \end{aligned}$ The second of the two compound inequalities is equivalent to $\left(x+3\right)\left(x-2\right) < 0$, as shown below. The solution set of $2x-3 < -x^2+x+3$ is the interval $\left(-3,2\right)$. $\begin{aligned} 2x-3 &< -x^2+x+3 \\ 2x-3+x^2-x-3 &< 0 \\ x^2+x-6 &< 0 \\ \left(x+3\right)\left(x-2\right) &< 0 \end{aligned}$ The solution of the compound inequality is the intersection of the two solution sets. Since the intersection of intervals $\left(0,3\right)$ and $\left(-3,2\right)$ is the interval $\left(0,2\right)$, the correct answer is option A. **Alternative Solution:** Note that the solution set of the inequality in the problem consists of all the values of x for which the graph of $y=3+x-x^2$ is above the graph of $y=\lvert 2x-3\rvert$. As it can be seen in the xy-plane below, the graph of $y=3+x-x^2$ is above the

Question Number	Correct Answer	Rationale
		graph of $y = \|2x - 3\|$ for all values of x greater than 0 and less than 2. Therefore, $0 < x < 2$, and the correct answer is option A.  ***Back to Question***

Question Number	Correct Answer	Rationale
7	C	**Option C is correct.** The question states that the fare is \$2.00 for the first quarter mile or less and \$0.75 for each quarter mile after that. Notice that all the options include a constant term of 2.00 (for the \$2.00 for the first quarter mile). Thus, the task is to model the fare for the remaining distance beyond the first quarter mile. Since the question states that \$0.75 is charged for each quarter mile after the first, one must determine how many quarter miles there are in the ride. Since the trip is m miles, where m is an integer, the number of quarter miles in the trip would be $4m$. The charge for the first quarter mile is \$2.00, so that would leave $4m - 1$ quarter miles to be charged at a rate of \$0.75 each. The total fare for the trip would thus be modeled by the equation $f = 2.00 + 0.75\ (4m - 1)$, and therefore the correct answer is option C. ***Back to Question***

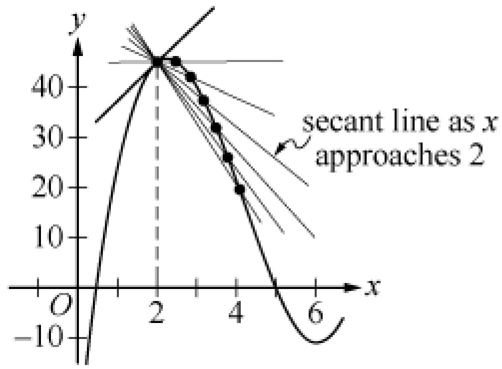
Question Number	Correct Answer	Rationale
8	C	**Option C is correct.** To find the x-coordinate of the point where the line with equation $6x + 3y = 24$ intersects the x-axis, set $y = 0$ in the equation $6x + 3y = 24$ and solve for x. The coordinates of the x-intercept of the line are $(4,0)$, as shown below. $\begin{aligned} 6x+3y&=24 \\ 6x+3(0)&=24 \\ 6x&=24 \\ x&=4 \end{aligned}$ To find the y-coordinate of the point where the line with equation $6x + 3y = 24$ intersects the y-axis, set $x = 0$ in the equation $6x + 3y = 24$ and solve for y. The coordinates of the y-intercept of the line are $(0,8)$, as shown below. $\begin{aligned} 6x+3y&=24 \\ 6(0)+3y&=24 \\ 3y&=24 \\ y&=8 \end{aligned}$ Therefore, the correct answer is option C. ***Back to Question***

Question Number	Correct Answer	Rationale
9	C	**Option C is correct.** The curve appears to be approximately linear between $x = 1$ and $x = 6$. The value of y at $x = 1$ is 10. The value of y at $x = 6$ is approximately 3.5. To approximate the average rate of change, in gallons per hour, over the interval $1 \leq x \leq 6$, use the slope of the line connecting the point with coordinates (1,10) to the point with coordinates (6,3.5). The slope of the line is $\frac{3.5-10}{6-1}$, which is approximately -1.30. Of the four options, the closest option to -1.30 is -1.20 , and therefore the correct answer is option C. ***Back to Question***
10	A	**Option A is correct.** Note that in all the options except option D, $f(-4) = f(-2) = f(3) = 0$, so option D is not correct. Option C is also not correct because the graph has two local extrema, and so it is the graph of a third-degree polynomial. In option B, the graph has four local extrema, which means it is the graph of a fifth-degree polynomial (with a double root at $x = -4$ and single roots at $x = -2$, $x = 0$, and $x = 3$). Option B also violates the “no other zeros” condition because it has a zero at $x = 0$. That leaves option A, which satisfies all the conditions. Option A is the graph of a fourth-degree polynomial with a double root at $x = -4$ and single roots at $x = -2$ and $x = 3$. Therefore, the correct answer is option A. ***Back to Question***

Question Number	Correct Answer	Rationale
11	C	**Option C is correct.** Since each of the x bracelets costs \$8 and each of the y necklaces costs \$12, Melissa's total income from the sale is $8x + 12y$ dollars. To reach her goal, Melissa's total income from the sale must be at least \$384. So, $8x + 12y \geq 384$, or equivalently $2x + 3y \geq 96$. The graph of the inequality $2x + 3y \geq 96$ in the xy-plane consists of all the points that are on or above the line $2x + 3y = 96$. The graph of the equation $2x + 3y = 96$ in the xy-plane is the line that connects the points (0,32) and (48,0). Since x and y represent numbers of objects, each of x and y must be a nonnegative number, and so the point with coordinates (x,y) must be in the first quadrant of the xy-plane (including the positive x-axis and the positive y-axis). The number of bracelets and necklaces that Melissa must sell in order to reach her goal is represented by all points in the first quadrant that are on or above the line $2x + 3y = 96$. The only option that shows a solid line from points (0,32) to (48,0) and a shaded area above the line is option C. ***Back to Question***
12	−28	**The correct answer is −28.** Use the recursive definition and the initial condition to compute the value of $f(5)$ as follows. $f(1) = 2$ $f(2) = 2f(1) - 4 = 2(2) - 4 = 0$ $f(3) = 2f(2) - 4 = 2(0) - 4 = -4$ $f(4) = 2f(3) - 4 = 2(-4) - 4 = -12$ $f(5) = 2f(4) - 4 = 2(-12) - 4 = -28$ Therefore, the correct answer is −28. ***Back to Question***

Question Number	Correct Answer	Rationale
13	B	**Option B is correct.** Recall that the first derivative of the function at a point is equal to the slope of the tangent line to the graph of the function at that point. By inspection, notice that starting near $x = 0$, the slope of the tangent line to the graph of $f(x)$ is negative and becomes less negative as x approaches a, and that the slope of the tangent line is 0 at $x = a$ (at the minimum value of f) and then becomes increasingly positive as x increases. Only option B is consistent with this behavior, and therefore the correct answer is option B. ***Back to Question***
14	B	**Option B is correct.** The instantaneous rate of change in the number of grams of substance produced 30 minutes after the reaction begins is the value of the first derivative of N evaluated at 30 minutes. First, convert 30 minutes into hours, then evaluate the first derivative of N at that value of t. Since 30 minutes equals $\frac{1}{2}$ hour, you will need to evaluate $N'\left(\frac{1}{2}\right)$. First find $N'(t)$. $$N'(t) = 16 - 8t$$ Therefore, $N'\left(\frac{1}{2}\right) = 16 - 8\left(\frac{1}{2}\right) = 12$. The answer is 12 grams per hour, option B. ***Back to Question***

Question Number	Correct Answer	Rationale
15	C	**Option C is correct.** The strength of the beam, *S*, is modeled by $S(w)=kw\left(a^2-w^2\right)$, or simply $S(w)=k\left(a^2w-w^3\right)$, where *k* is a positive constant. The derivative of $S(w)$ with respect to *w* is $S'(w)=k\left(a^2-3w^2\right)$. To find the maximum strength of the beam, solve the equation $S'(w)=0$ to find the critical point, and then use the first-derivative test to confirm that the critical point is a global maximum. Since $k\left(a^2-3w^2\right)=0$ and *k* is positive, it follows that $a^2-3w^2=0$, or $a^2=3w^2$. Since *w* is positive because it represents a distance, it follows that the critical point is $w=\frac{a}{\sqrt{3}}$. Note that $S'(w)=k\left(a^2-3w^2\right)$ is positive for $0<w<\frac{a}{\sqrt{3}}$ and negative for $w>\frac{a}{\sqrt{3}}$. Using the first-derivative test, it follows that the critical point $w=\frac{a}{\sqrt{3}}$ is a global maximum of the function *S*. The Pythagorean theorem applied to the triangle with sides *d* and *w* and hypotenuse *a* gives $a^2=d^2+w^2$. Substituting $a^2=3w^2$ in $a^2=d^2+w^2$ produces $3w^2=d^2+w^2$, or $d^2=2w^2$. Therefore, $d=\sqrt{2}w$, and the correct answer is option C. ***Back to Question***

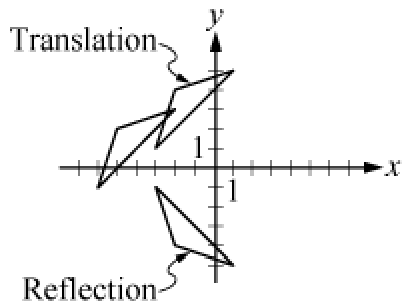
Question Number	Correct Answer	Rationale
16	B	**Option B is correct.** As shown in the figure below, as x approaches 2, the slope of the secant line that passes through the points $(x, f(x))$ and $(2, f(2))$ approaches the slope of $f(x)$ at $(2, f(2))$. That is, it approaches $f'(2)$, the derivative of f at $x = 2$.  Since $f(x) = 2x^3 - 25x^2 + 82x - 35$, the derivative of f is $f'(x) = 6x^2 - 50x + 82$. At $x = 2$, the derivative of f is $f'(2) = 6$, and therefore the correct answer is option B. ***Back to Question***

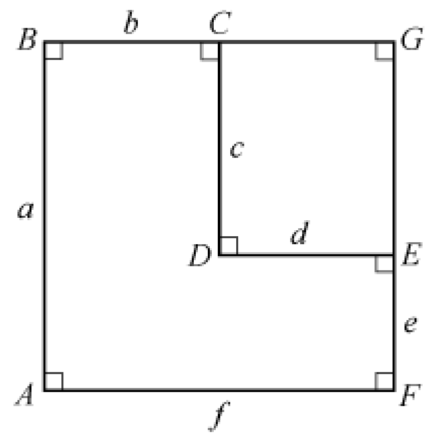
Question Number	Correct Answer	Rationale
17	B	**Option B is correct.** Since the graph of function g is below the graph of function f for all values of x, then $ax^2 + b < 3x^2$ for all values of x; that is, $(a - 3)x^2 + b < 0$ for all values of x. In particular, substituting 0 for x in the last inequality gives $(a - 3)(0) + b < 0$, or $b < 0$, so b is negative. If a were greater than 3, then the graph of $y = (a - 3)x^2 + b$ would be a parabola that opens upward and there would be values of x that would make $(a - 3)x^2 + b$ positive, which contradicts the information that $(a - 3)x^2 + b < 0$ for all values of x; this contradiction means that $a \leq 3$ must be true. Therefore, the correct answer is option B. ***Back to Question***
18	D	**Option D is correct.** The question asks for a verbal description of the change in the rabbit population, based on the function given. Recall the meaning of the base (growth factor) and the exponent in an exponential growth model. Note that the function given $P(t) = 250 \cdot (3.04)^{\frac{t}{1.98}} \approx 250 \cdot 3^{\frac{t}{2}}$. Observe from this approximation, with base 3 and exponent $\frac{t}{2}$, that the population tripled every two years. In fact, $(3.04)^{\frac{2}{1.98}} \approx 3.07$, so the population tripled in a time period of a little less than 2 years. Thus, the correct answer is option D, “The rabbit population tripled every 24 months.” ***Back to Question***

Question Number	Correct Answer	Rationale
19	B	**Option B is correct.** By using a graphing calculator, one can find that the coordinates of point A, the intersection of the graphs of $f(x) = -x$ and $g(x) = e^x$, are approximately $(-0.568, 0.568)$. Point A is also on the graph of $h(x) = bx - 1$. Another point on the graph of $h(x) = bx - 1$ is $(0, -1)$. Use these two points to find the value of b, which is the slope of the graph of $h(x)$, as shown below. $$\begin{aligned} b &= \frac{0.568 - (-1)}{-0.568 - 0} \\ &= \frac{1.568}{-0.568} \\ &\approx -2.76 \end{aligned}$$ Therefore, the correct answer is option B. Note: If $(-0.57, 0.57)$ are used as the coordinates of point A, the answer is $b \approx -2.75$, which is even closer to option B. ***Back to Question***

Question Number	Correct Answer	Rationale
20	B	**Option B is correct.** Since $f(x)=2x+4$ and $g(x)=\frac{x}{2}+b$ for all numbers *x*, it follows that $(f\circ g)(x)=x+2b+4$ and $(g\circ f)(x)=x+2+b$ for all numbers *x*, as shown below. $$\begin{aligned}(f\circ g)(x)&=f(g(x))\\&=2(g(x))+4\\&=2\left(\frac{x}{2}+b\right)+4\\&=x+2b+4\end{aligned}$$ $$\begin{aligned}(g\circ f)(x)&=g(f(x))\\&=\frac{f(x)}{2}+b\\&=\frac{2x+4}{2}+b\\&=x+2+b\end{aligned}$$ Given that $(f\circ g)(x)=(g\circ f)(x)$ for all numbers *x*, it follows that $b=-2$, as shown below. $$\begin{aligned}x+2b+4&=x+2+b\\2b+4&=2+b\\b&=-4+2\\b&=-2\end{aligned}$$ Therefore, the correct answer is option B. ***Back to Question***

Question Number	Correct Answer	Rationale
21	C	**Option C is correct.** The domain of the function *f* consists of all real numbers *x* such that $\sqrt{x^2-4}$ is a real number. If x^2-4 were negative, then $\sqrt{x^2-4}$ would not be a real number. So, $x^2-4\ge 0$, or equivalently $x^2\ge 4$, which means that the domain of *f* consists of all real numbers *x* that are less than or equal to −2 or that are greater than or equal to 2. Therefore, the correct answer is option C. ***Back to Question***
22	C	**Option C is correct.** If 3 students shared the cost of *n* dollars equally, then the cost per student would be $\frac{n}{3}$ dollars. If 5 students shared the cost equally, then the cost per student would be $\frac{n}{5}$ dollars. It is given that the cost per student when 3 students share is *d* dollars greater than the cost per student when 5 students share. So, $\frac{n}{3}-\frac{n}{5}=d$, which is equivalent to $n=7.5d$, as shown below. $$\begin{aligned}\frac{n}{3}-\frac{n}{5}&=d\\ \frac{5n-3n}{15}&=d\\ 2n&=15d\\ n&=\frac{15d}{2}\\ n&=7.5d\end{aligned}$$ Therefore, the correct answer is option C. ***Back to Question***

Question Number	Correct Answer	Rationale
23	A	**Option A is correct.** To do the translation, move each vertex of the original triangle 3 units to the right, then 2 units up, and finally connect the 3 translated vertices. That is, the vertex at $(-6,-1)$ moves to $(-3,1)$, and so on. (See translated triangle below.) Then, to reflect the translated triangle across the *x*-axis, move each vertex to a point that is as many units below the *x*-axis as it is above the *x*-axis. That is, the vertex at $(-3,1)$, which is 1 unit above the *x*-axis, moves to $(-3,-1)$, which is 1 unit below the *x*-axis, and so on. (See reflected triangle below.) Therefore, the correct answer is option A.  ***Back to Question***

Question Number	Correct Answer	Rationale
24	A, B, C	**Options A, B, and C are correct.** Consider the figure below, where the vertices have been labeled and the line segments $\overline{BC}$ and $\overline{FE}$ have been extended until they intersect at point G to form rectangle $ABGF$. Consider each option.  Option A: Since $CDEG$ is a rectangle, $CG = DE = d$. Since $ABGF$ is a rectangle, $BG = AF = f$. Since $BG = BC + CG = b + d$, one can conclude that $f = b + d$. Option B: The perimeter is $P = a + (c + e) + f + (b + d)$. As shown above, $b + d = f$. Similarly, one can argue that $c + e = a$. So, the perimeter is $P = a + a + f + f = 2a + 2f$. Option C: The area of the region is the area of rectangle $ABGF$ (which is the product af) minus the area of rectangle $CDEG$ (which is the product cd). That is, the area is $A = af - cd$. Therefore, the correct answers are options A, B, and C. ***Back to Question***

Question Number	Correct Answer	Rationale
25	D	**Option D is correct.** Recall that the Law of Sines relates the lengths of two sides of a triangle and the sines of the angles opposite the sides. (The Law of Sines is included in the Notations, Definitions, and Formulas pages that are included in this Study Companion and at the beginning of the test.) Using the Law of Sines yields $\frac{\sin(\angle BAC)}{\sin(\angle BCA)} = \frac{BC}{BA}$ and $\frac{\sin 30°}{\sin(\angle BCA)} = \frac{9}{12}$. Therefore, $\sin(\angle BCA) = \frac{4}{3}\sin 30° = \frac{2}{3}$. Recall that this is an example of the ambiguous case of the Law of Sines since the value of the sine is between 0 and 1. There are two angles between 0 degrees and 180 degrees, one acute and one obtuse, associated with this sine, and therefore there are two possible triangles with the given sides and angle measure. The correct answer, therefore, is option D, "It cannot be determined from the information given." The two values of the measure of $\angle BCA$ are approximately 41.8° and 138.2°. Using either the Law of Sines again (with $\angle BAC$ and $\angle ABC$, or with $\angle BCA$ and $\angle ABC$) or the Law of Cosines, it can be determined that the length of side $\overline{AC}$ is either approximately 3.68 or 17.10. Since the length of side $\overline{AC}$ cannot be uniquely determined, the correct answer is option D. ***Back to Question***

Question Number	Correct Answer	Rationale
26	B	**Option B is correct.** To begin, consider a single point on the octagon, such as the point $(0,4)$ at the top of the octagon in the figure. This point is 4 units from the origin, so any rotation that maps the octagon onto itself would need to map this point onto a point that is also 4 units from the origin. The only other point on the octagon that is 4 units from the origin is the point $(0,-4)$. A rotation of angle $\theta = \pi$ would map the point $(0,4)$ onto the point $(0,-4)$. The octagon is symmetric about the *x*- and *y*-axes, so a rotation of angle $\theta = \pi$ would map all of the points of the octagon onto corresponding points of the octagon. Likewise, a rotation of angle $\theta = 2\pi$ would map the point $(0,4)$ onto itself (and map all other points of the octagon onto themselves). No other values of θ such that $0 < \theta \leq 2\pi$ would map the octagon onto itself. Therefore the correct answer is two, option B. ***Back to Question***

Question Number	Correct Answer	Rationale
27	C	**Option C is correct.** To determine the length of $\overline{BC}$, it would be helpful to first label the figure with the information given. Since the circle has radius 2, then both $\overline{OC}$ and $\overline{OP}$ have length 2 and $\overline{CP}$ has length 4. $\overline{AP}$ is tangent to the circle at *P*, so angle *APC* is a right angle. The length of $\overline{AP}$ is given as 3. This means that triangle *ACP* is a 3-4-5 right triangle and $\overline{AC}$ has length 5. Notice that since $\overline{CP}$ is a diameter of the circle, angle *CBP* is also a right angle. Angle *BCP* is in both triangle *ACP* and triangle *PCB*, and, therefore, the two triangles are similar. Then find the length of $\overline{BC}$ by setting up a proportion between the corresponding parts of the similar triangles as follows: $$\frac{CP}{AC}=\frac{BC}{PC}$$ $$\frac{4}{5}=\frac{BC}{4}$$ $$BC=\frac{16}{5}=3.2$$ The correct answer, 3.2, is option C. ***Back to Question***

Question Number	Correct Answer	Rationale
28	D	**Option D is correct.** The data in a stem-and-leaf plot is ordered, so finding the median, the middle number when the data are ordered from least to greatest or greatest to least, is straightforward. Given the course grades received by 22 students, the median course grade would be the average of the course grades of the 11th and 12th students. Start at either the least or greatest data entry and count in increasing (or decreasing) order along the leaves until you reach the 11th and 12th entries. In this case, both the 11th and 12th entries have a value of 82 (i.e., a stem value of 8 and a leaf value of 2). Therefore the median course grade received by the 22 students is 82. The correct answer is option D. ***Back to Question***
29	D	**Option D is correct.** Consider each option. Option A: The boxplot shows that 29 minutes is the median of the 20 times, so half of the times have values from 18 to 29 minutes, while the values of the other half are distributed from 29 to 45 minutes, which makes it very likely that the mean of the 20 times is greater than 29 minutes. So, option A is not correct. Option B: The number of runners who finished in a time between 35 and 45 minutes is 25 percent of 20, and the number who finished in a time between 18 and 22 minutes is also 25 percent of 20. So, option B is not correct. Option C: The number of runners who finished in a time between 22 and 35 minutes is 50 percent (or half) of 20, not more than half. So, option C is not correct. Option D: The numbers at each end of the boxplot indicate that at least one runner finished in 18 minutes and at least one runner finished in 45 minutes. Therefore, the correct answer is option D. ***Back to Question***

Question Number	Correct Answer	Rationale
30	D	**Option D is correct.** To find a person's standardized score (also known as the *z*-score) on a test, subtract the test's mean from the person's score and then divide the result by the test's standard deviation. The table provides all the information needed to find Martha's standardized score on each of the four tests. Consider each option. Option A: Martha's standardized score on the English test is $\frac{74-76}{5.0}$, or −0.40. Option B: Martha's standardized score on the math test is $\frac{73-63}{6.0}$, or approximately 1.67. Option C: Martha's standardized score on the history test is $\frac{63-58}{5.5}$, or approximately 0.91. Option D: Martha's standardized score on the biology test is $\frac{69-60}{4.5}$, or 2.00. So Martha's highest standardized score is 2.00 on the biology test. Therefore, the correct answer is option D. ***Back to Question***

Question Number	Correct Answer	Rationale
31	C	**Option C is correct.** Recall that approximately 68% of a normally distributed set of data lie within ±1 standard deviation of the mean and approximately 95% of the data lie within ±2 standard deviations of the mean. Evaluate each option in order to determine which of the groups has the greatest percent. Option A: Since the mean handspan is 7 inches and the standard deviation is 1 inch, the group of handspan measures that is less than 6 inches is the group that is more than 1 standard deviation less than the mean. The group of handspan measures that is less than 7 inches includes 50% of the measures. Approximately 34 percent ($\frac{1}{2}$ of 68 percent) of the measures are between 6 inches and 7 inches (within 1 standard deviation less than the mean). So, the group with handspan measures less than 6 inches would be approximately equal to 50% − 34%, or 16% of the measures. Option B: Since 7 inches is the mean, approximately 50% of the measures are greater than the mean. Option C: This is the group that is within ±1 standard deviation of the mean. This group contains approximately 68% of the measures. Option D: This group is between the mean and 2 standard deviations less than the mean. Approximately 47.5% ($\frac{1}{2}$ of 95%) of the measures are between 5 inches and 7 inches. Of the options given, the group described in option C is expected to contain the greatest percent of the measures, approximately 68%, so option C is the correct answer. ***Back to Question***

Question Number	Correct Answer	Rationale
32	A	**Option A is correct.** Because each toss of the coin is an independent event, the probability of tossing heads then 2 tails, $P(HTT)$, is equal to $P(H) \cdot P(T) \cdot P(T)$, where $P(H)$ is the probability of tossing heads and $P(T)$ is the probability of tossing tails. The probability of tossing heads is twice the probability of tossing tails, so $P(H) = \frac{2}{3}$ and $P(T) = \frac{1}{3}$. Therefore, $P(HTT) = \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{2}{27}$. There are 3 ways in which exactly 2 of 3 tosses would be tails, and each of them has an equal probability of occurring: $P(THT) = P(TTH) = P(HTT) = \frac{2}{27}$. Therefore, the total probability that tails will result exactly 2 times in 3 tosses is $3\left(\frac{2}{27}\right) = \frac{2}{9}$. So, the correct answer is option A. ***Back to Question***

Preparation Resources

The resources listed below may help you prepare for the GACE assessment in this field. These preparation resources have been identified by content experts in the field to provide up-to-date information that relates to the field in general. You may wish to use current issues or editions of these materials to obtain information on specific topics for study and review.

Calculator Use

An on-screen calculator is part of the testing software for this assessment.
A free tutorial is available so you can practice using the calculator prior to taking the assessment. Access the tutorial in Test Preparation section of the GACE website at ***www.gace.ets.org/prepare/tutorials/calculator***.

Guide to Taking a GACE Computer-delivered Assessment

This guide explains how to navigate through a GACE assessment and how to answer different types of test questions. This free download is available in the Test Preparation Resources section of the GACE website at ***www.gace.ets.org/prepare***.

Reducing Test Anxiety

This guide provides practical help for people who suffer from test anxiety. Designed specifically for GACE test takers, but useful to anyone who has to take tests, this guide reviews the major causes of test anxiety and offers practical advice for how to counter each one. Download this guide for free from the Test Preparation Resources section of the GACE website at ***www.gace.ets.org/prepare***.

Study Tips: Preparing for a GACE Assessment

This document contains useful information on preparing for selected-response and constructed-response tests. The instruction, tips, and suggestions can help you become a better-prepared test taker. See the Test Preparation Resources section of the GACE website at ***www.gace.ets.org/prepare*** for this free download.

Journals

Mathematics Teacher, National Council of Teachers of Mathematics

Mathematics Teaching in the Middle School, National Council of Teachers of Mathematics

Middle Ground, National Middle School Association

Middle School Journal, National Middle School Association

Other Resources

Ball, D. L. (1992). *Magical Hopes: Manipulatives and the Reform of Math Education.* American Educator; v16 n2, 14–18, 46–47.

Bittenger, M. L., and Ellenbogen, D. (1997). *Elementary Algebra: Concepts and Applications,* Fifth Edition. Menlo Park, Calif.: Addison-Welsey.

Brahier, D. J. (1999). *Teaching Secondary and Middle School Mathematics*. Needham Heights, Mass,: Allyn and Bacon.

Burns, M. (2000). *About Teaching Mathematics: A K–8 Resource,* Second Edition. Sausalito, Calif.: Math Solutions Publications.

Chapin, S. H., et al. (1999). *Middle Grades Math: Tools for Success. Course 3*. Needham, Mass.: Prentice Hall.

Coxford, A., Usiskin, Z., and Hirschhorn, D. (1998). *The University of Chicago School of Mathematics Project: Geometry.* Glenview, Ill.: Scott, Foresman and Company.

Crouse, R. J., and Sloyer, C. W. (1987). *Mathematical Questions from the Classroom—Parts I and II.* Providence, R.I.: Janson Publications.

Danielson, C. (1997). *A Collection of Performance Tasks and Rubrics: Middle School Mathematics*. Larchmont, N.Y.: Eye On Education, Inc.

Demana, F., Waits, B. K., Clemens, S. R., and Foley, G. D. (1997). *Precalculus: A Graphing Approach,* Fourth Edition. Menlo Park, Calif.: Addison-Wesley.

Foerster, P. A. (1998). *Calculus Concepts and Applications*. Berkeley, Calif.: Key Curriculum Press.

Garcia, J., Spaulding, E., and Powell, E. R. (2001). *Contexts of Teaching: Methods for Middle and High School Instruction,* First Edition. Upper Saddle River, N.J.: Prentice Hall.

Gottlieb, R. J. (2001). *Calculus: An Integrated Approach to Functions and Their Rates of Change,* Preliminary Edition. Boston, Mass.: Addison Wesley Longman, Inc.

Harshbarger, R. J., and Reynolds, J. J. (1992). *Mathematical Applications for the Management, Life, and Social Sciences,* Fourth Edition. Lexington, Mass.: D. C. Heath and Company.

Hungerford, T. W. (2001). *Contemporary College Algebra and Trigonometry: A Graphing Approach*. Philadelphia, Pa.: Harcourt College Publishers.

Jackson, A. W., and Davis, G. A. (2000). *Turning Points 2000: Educating Adolescents in the 21st Century*. New York, N.Y.: Carnegie Corporation of New York.

Jaworski, B., and Watson, A. (1994). *Mentoring in Mathematics Teaching*. London: The Felma Press.

Kilpatrick, J., Swafford, J., and Finell, B. (Eds.). (2001). *Adding It Up: Helping Children Learn Mathematics.* Washington, District of Columbia: National Academy Press.

Knowles, T., and Brown, D. F. (2000). *What Every Middle School Teacher Should Know*. Westerville, Ohio: National Middle School Association.

Leitzel, J. R. C. (1991). *A Call for Change: Recommendations for the Mathematical Preparation of Teachers of Mathematics*. Washington, District of Columbia: Mathematics Association of America.

Ma, L. (1999). *Knowing and Teaching Elementary Mathematics: Teachers' Understanding of Fundamental Mathematics in China and the United States.* Mahwah, N.J.: Lawrence Erlbaum.

Merenbloom, E. Y. (1991). *The Team Process: A Handbook for Teachers*. Columbus, Ohio: National Middle School Association.

Morrow, L. J., and Kenney, M. J. (Eds.). (1998). *The Teaching and Learning of Algorithms in School Mathematics.* Reston, Va.: The National Council of Teachers of Mathematics, Inc.

National Council of Teachers of Mathematics. (1995). *Assessment Standards for School Mathematics.* Reston, Va.: The National Council of Teachers of Mathematics, Inc.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, Va.: The National Council of Teachers of Mathematics, Inc.

Newmark, J. (1997). *Statistics and Probability in Modern Life* (6th ed.). Philadelphia, Pa.: Saunders College Publishing.

Ostebee, A., and Zorn, P. (1997). *Calculus from Graphical, Numerical, and Symbolic Points of View.* Philadelphia, Pa.: Harcourt College Publishers.

Peters, K. H., and March, J. K. (1999). *Collaborative Observation. Thousand Oaks*, Calif.: Corwin Press.

Schultze, J. E., Kennedy, P. A., et al. (2001). *Algebra 1*. Austin, Texas: Holt, Rinehart, and Winston.

Schultze, J. E., Ellis, W., Jr., et al. (2001). *Algebra 2*. Austin, Texas: Holt, Rinehart, and Winston.

Schultze, J. E., Hollowell, K. A., et al. (2001). *Geometry*. Austin, Texas: Holt, Rinehart, and Winston.

Serra, M. (1997). *Discovering Geometry: An Inductive Approach,* Second Edition. Emeryville, Calif.: Key Curriculum Press.

Stein, M., Silbert, J., and Carnine, D. (1997). *Designing Effective Mathematics Instruction: A Direct Instruction Approach.* Upper Saddle River, N.J.: Merrill Prentice Hall.

Swanson, T., Andersen, J., and Keeley, R. (2000). *Precalculus: A Study of Functions and Their Application*. Fort Worth, Texas: Harcourt College Publishers.

Swokowski, E. W., Olinick, M., and Pence, D. D. (1994). *Calculus of a Single Variable,* Second Edition. Stamford, Conn.: Brooks/Cole.

Triola, M. F. (2001). *Elementary Statistics,* Eighth Edition. Boston, Mass.: Addison Wesley Longman, Inc.

Wallace, E. C., and West, S. F. (1998). *Roads to Geometry*, Second Edition. Upper Saddle River, N.J.: Prentice Hall.

Williams, G. (2000). *Applied College Algebra: A Graphing Approach*. Philadelphia, Pa.: Harcourt College Publishers.

Wu, H. (1999). *Basic Skills Versus Conceptual Understanding: A Bogus Dichotomy in Mathematics Education*. American Educator; v23 n3, 14–19, 50–52.

Online Resources

Georgia Department of Education — ***www.doe.k12.ga.us***

Mathematics TEKS Toolkit, The Charles A. Dana Center at the University of Texas at Austin — ***www.utdanacenter.org/mathtoolkit***

National Council of Teachers of Mathematics — ***www.nctm.org***